# QUASI-UNIFORMITY OF MINIMAL WEIGHTED ENERGY POINTS ON COMPACT METRIC SPACES

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Abstract. For a closed subset $K$ of a compact metric space $A$ possessing an $\alpha$-regular measure $\mu$ with $\mu(K) > 0$, we prove that whenever $s > \alpha$, any sequence of weighted minimal Riesz $s$-energy configurations $\omega_N = \{x_{i,N}^{(s)}\}_{i=1}^N$ on $K$ (for 'nice' weights) is quasi-uniform in the sense that the ratios of its mesh norm to separation distance remain bounded as $N$ grows large. Furthermore, if $K$ is an $\alpha$-rectifiable compact subset of Euclidean space ($\alpha$ an integer) with positive and finite $\alpha$-dimensional Hausdorff measure, it is possible to generate such a quasi-uniform sequence of configurations that also has (as $N \to \infty$) a prescribed positive continuous limit distribution with respect to $\alpha$-dimensional Hausdorff measure.

## 1. Introduction

Let $A$ be a compact infinite metric space with metric $:A \times A \to [0, \infty)$ and let $\omega_N = \{x_i\}_{i=1}^N \subset A$ denote a configuration of $N \geq 2$ points in $A$. We are chiefly concerned with two 'quality' measures of $\omega_N$; namely, the *separation distance of* $\omega_N$ defined by

$$\delta(\omega_N) := \min_{1 \leq i \neq j \leq N} (x_i, x_j), \tag{1.1}$$

and the *mesh norm of* $\omega_N$ *with respect to* $A$ defined by

$$\rho(\omega_N, A) := \max_{y \in A} \min_{1 \leq i \leq N} (y, x_i). \tag{1.2}$$

This quantity is also known as the *fill radius* or *covering radius* of $\omega_N$ relative to $A$. The optimal values of these quantities are also of interest and we consider, for $N \geq 2$, the *$N$-point best-packing distance on* $A$ given by

$$\delta_N(A) := \max\{\delta(\omega_N) \colon \omega_N \subset A,\, |\omega_N| = N\},$$

and the *$N$-point mesh norm* of $A$ given by

$$\rho_N(A) := \min\{\rho(\omega_N, A) \colon \omega_N \subset A,\, |\omega_N| = N\},$$

where $|S|$ denotes the cardinality of set $S$.

In the theory of approximation and interpolation (for example, by splines or radial basis functions (RBFs)), the separation distance is often associated with some

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measure of 'stability' of the approximation, while the mesh norm arises in the error of the approximation. In this context, the *mesh-separation ratio* (or *mesh ratio*)

$$\gamma(\omega_N, A) := \rho(\omega_N, A)/\delta(\omega_N),$$

can be regarded as a 'condition number' for $\omega_N$ relative to $A$. If $\{\omega_N\}_{N=2}^{\infty}$ is a sequence of $N$-point configurations such that $\gamma(\omega_N, A)$ is uniformly bounded in $N$, then the sequence is said to be *quasi-uniform on* $A$. Quasi-uniform sequences of configurations are important for a number of methods involving RBF approximation and interpolation (see [9, 15, 17, 19]).

We remark that in some cases it is easy to obtain positive lower bounds for the mesh-separation ratio. For example, if $A$ is connected, then $\gamma(\omega_N, A) \geq 1/2$. Furthermore, letting

$$B(x,r) = \{y \in A : m(y,x) \leq r\}$$

be the closed ball in $A$ with center $x$ and radius $r$, then $\gamma(\omega_N, A) \geq \beta/2$ for any $N$-point configuration $\omega_N \subset A$ whenever $A$ and $\beta \in (0,1)$ have the property that for any $r \in (0, \mathrm{diam}(A)]$ and any $x \in A$, the annulus $B(x,r) \setminus B(x, \beta r)$ is nonempty. The diameter of $A$ is defined by

$$\mathrm{diam}(A) := \max\{m(x,y) \colon x \in A,\ y \in A\}.$$

In this paper we consider the separation distance and mesh norm of finite point configurations in $A$ that minimize certain weighted energy functionals. We call $w : A \times A \to [0, \infty)$ an *SLP weight on* $A$ if it is symmetric and lower semi-continuous on $A \times A$ and is positive on the diagonal, $D(A)$, of $A \times A$. For $s > 0$ and a collection of $N \geq 2$ distinct points $\omega_N = \{x_1, \ldots, x_N\} \subset A$, the $(s,w)$*-energy of* $\omega_N$ (also known as the *weighted Riesz* $s$*-energy*) is

$$E_s^w(\omega_N) := \sum_{i \neq j} \frac{w(x_i, x_j)}{(x_i, x_j)^s} = \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{w(x_i, x_j)}{(x_i, x_j)^s}, \tag{1.3}$$

and we denote the *minimal* $N$*-point* $(s,w)$*-energy of* $A$ by

$$\mathcal{E}_s^w(N, A) := \inf\{E_s^w(\omega_N) : \omega_N \subset A,\ |\omega_N| = N\}. \tag{1.4}$$

Since $A$ is compact and the energy $E_s^w(\omega_N)$ is lower semi-continuous, there exists at least one $N$-point configuration $\omega_N^* \subset A$ such that $E_s^w(\omega_N^*) = \mathcal{E}_s^w(N, A)$. We refer to such an $\omega_N^*$ as an $N$*-point* $(s,w)$*-energy minimizing configuration on* $A$. The asymptotics as $N \to \infty$ of $N$-point $(s,w)$-energy minimizing configurations and their energies are investigated in [2, 10] for $d$-rectifiable sets $A \subset \mathbb{R}^p$ and $s > d$ (see further discussion in the next section).

In our results we shall require that $A$ is either $\alpha$-regular or upper $\alpha$-regular as we next describe. For a positive Borel measure $\mu$ supported on $A$ and $\alpha > 0$, we say that $\mu$ is *upper* $\alpha$*-regular* if there is some finite constant $C_0$ such that

$$\mu(B(x,r)) \leq C_0\, r^{\alpha} \qquad (x \in A,\ 0 < r \leq \mathrm{diam}(A)), \tag{1.5}$$

and we say that $\mu$ is *lower* $\alpha$*-regular* if there is some positive constant $c_0$ such that

$$c_0^{-1}\, r^{\alpha} \leq \mu(B(x,r)) \qquad (x \in A,\ 0 < r \leq \mathrm{diam}(A)). \tag{1.6}$$

We shall refer to $A$ as an *upper* $\alpha$*-regular metric space* if there exists an upper $\alpha$-regular measure $\bar{\mu}$ on $A$ such that $\bar{\mu}(A) > 0$ and shall refer to $A$ as a *lower* $\alpha$*-regular*

*metric space* if there exists a lower $\alpha$-regular measure $\underline{\mu}$ on $A$ such that $\underline{\mu}(A) < \infty$. (Obviously, if $A$ is upper $\alpha$-regular then $A$ has infinitely many points.) If $A$ supports a measure that is both upper and lower $\alpha$-regular, then we say that $A$ is an *$\alpha$-regular metric space.* If $A$ is $\alpha$-regular, then it is not difficult to show that the Hausdorff dimension of $A$, $\dim_{\mathcal{H}} A$, equals $\alpha$ (cf. [12, 16]). Furthermore, the $\alpha$-dimensional Hausdorff measure of $A$, $\mathcal{H}_\alpha(A)$, is positive and finite.

Many of the constants appearing in this paper, either explicitly or implicitly involve the upper and lower regularity constants $C_0$ and $c_0$ appearing in (1.5) and (1.6). However, in certain cases we are interested in 'local' regularity estimates (i.e., for $r$ small) which can substantially improve our explicit estimates for particular metric spaces of interest (e.g., $A$ is the sphere $S^d$ with the Euclidean metric). Specifically, if $\bar{\mu}$ is an upper $\alpha$-regular measure, $\underline{\mu}$ is a lower $\alpha$-regular measure and $r^* > 0$, we define

$$\begin{aligned} C_0(r^*) &:= \sup\{\bar{\mu}(B(x,r))/r^\alpha \colon x \in A,\ 0 < r \le r^*\}, \\ c_0(r^*)^{-1} &:= \inf\{\underline{\mu}(B(x,r))/r^\alpha \colon x \in A,\ 0 < r \le r^*\}. \end{aligned} \tag{1.7}$$

We note that both $C_0(r^*)$ and $c_0(r^*)$ are increasing in $r^*$, and we make the definitions

$$\begin{aligned} C_0(0) &:= \lim_{r^* \to 0^+} C_0(r^*), \\ c_0(0) &:= \lim_{r^* \to 0^+} c_0(r^*). \end{aligned} \tag{1.8}$$

Furthermore, if $A$ is a compact (i.e., without boundary), $C^1$, $d$-dimensional manifold and $\mu = \mathcal{H}_d$, then $C_0(0) \cdot c_0(0) = 1$. For the largest length scale of interest, with a slight abuse of notation, the global constants for $\bar{\mu}$ and $\underline{\mu}$, respectively, are related by $C_0 = C_0(\mathrm{diam}(A))$ and $c_0 = c_0(\mathrm{diam}(A))$.

One may obtain simple upper bounds for $\delta_N(A)$ (respectively, lower bounds for $\rho_N(A)$) in the case that $A$ is lower (respectively, upper) $\alpha$-regular. Specifically, if $A$ is lower $\alpha$-regular then there is a constant $c_A < \infty$ such that

$$\delta_N(A) \le c_A N^{-1/\alpha}, \qquad (N \ge 2), \tag{1.9}$$

while if $A$ is upper $\alpha$-regular then there is a constant $\tilde{c}_A > 0$ such that

$$\rho_N(A) \ge \tilde{c}_A N^{-1/\alpha}, \qquad (N \ge 2). \tag{1.10}$$

The bound (1.9) is a consequence of the facts that the balls $\{B(x, \delta(\omega_N)/2) \colon x \in \omega_N\}$ are pairwise disjoint and that there exists a lower $\alpha$-regular measure $\underline{\mu}$ with $\underline{\mu}(A) < \infty$. Similarly, if $A$ is upper $\alpha$-regular, then the bound (1.10) follows from the covering property of the balls $\{B(x, \rho(\omega_N, A)) \colon x \in \omega_N\}$ and the existence of an upper $\alpha$-regular measure $\bar{\mu}$ with $\bar{\mu}(A) > 0$.

The main result of this paper, given in Theorem 5, is that a sequence of $N$-point $(s,w)$-energy minimizing configurations on an $\alpha$-regular compact metric space $A$ is quasi-uniform on $A$ whenever $s > \alpha$. As an application, we deduce that, if $A \subset \mathbb{R}^p$ is $d$-rectifiable for some integer $0 < d \le p$ with $\mathcal{H}_d(A) > 0$, then a quasi-uniform sequence of $N$-point configurations on $A$ can be found that has a prescribed bounded positive density on $A$ (see Corollary 6 and the discussion preceding it).

## 2. Main Results

We first consider the separation distance of $(s, w)$-energy minimizing configurations on an upper $\alpha$-regular compact metric space $A$. For these separation results, we consider symmetric weight functions $w$ such that $\|w(\cdot, x)\|_{L_p(\mu)}$ is uniformly bounded on $A$ for some $1 < p \leq \infty$. Here we use the standard notation,

$$\|f\|_{L_p(\mu)} := \begin{cases} \left(\int_A |f|^p \, d\mu\right)^{1/p}, & 1 \leq p < \infty, \\ \mu\text{-ess sup } |f|, & p = \infty, \end{cases}$$

where $\mu$ is a positive Borel measure and $f$ is a Borel measurable function on $A$.

The following theorem extends a result [2, Theorem 4] to a more general class of weight functions and to more general compact metric spaces.

**Theorem 1.** *Let $A$ be a compact, upper $\alpha$-regular metric space with respect to $\bar{\mu}$ and let $w$ be an SLP weight on $A$ such that $\|w(\cdot, x)\|_{L_{p_0}(\bar{\mu})}$ is uniformly bounded on $A$ for some $1 < p_0 \leq \infty$. Suppose $1 < p \leq p_0$, $s > \alpha(1 - 1/p)$, and $N \geq 2$. If $\omega_N^*$ is an $N$-point $(s, w)$-energy minimizing configuration on $A$, then*

$$\delta(\omega_N^*) \geq C_1 \, N^{-\left(\frac{1}{\alpha} + \frac{1}{sp}\right)} \qquad (N \geq 2), \tag{2.1}$$

*where $C_1$ is a constant independent of $N$ indicated below in* (3.13).

Taking $w$ bounded and setting $p = \infty$ in Theorem 1 produces the following result.

**Corollary 2.** *Suppose $A$ is a compact, upper $\alpha$-regular metric space and $w$ is a bounded SLP weight on $A$, and let $s > \alpha$. If $\omega_N^*$ is an $N$-point $(s, w)$-energy minimizing configuration on $A$, then*

$$\delta(\omega_N^*) \geq C_2 N^{-1/\alpha} \qquad (N \geq 2), \tag{2.2}$$

*where $C_2$ is a constant independent of $N$. Consequently,*

$$\delta_N(A) \geq C_2 N^{-1/\alpha} \qquad (N \geq 2). \tag{2.3}$$

*For the unweighted case $w \equiv 1$, the constant $C_2$ satisfies*

$$C_2 \geq \left[\frac{\bar{\mu}(A)}{C_0}\left(1 - \frac{\alpha}{s}\right)\right]^{1/\alpha} \left(\frac{\alpha}{s}\right)^{1/s}, \tag{2.4}$$

*where $C_0 = C_0(\mathrm{diam}(A))$.*

We note that if $A$ in Corollary 2 is $\alpha$-regular, then by inequality (1.9) we see that $N$-point $(s, w)$-energy minimizing configurations on $A$ have the best possible order of separation as $N \to \infty$.

With respect to the separation constant of (2.4), if $d \geq 2$ and $A = \mathbb{S}^d$ with $\sigma_d$ denoting the uniform probability distribution on $\mathbb{S}^d$, then we can get an explicit lower bound for $C_2$ by calculating the regularity constant $C_0$. As stated in [13], for $x \in \mathbb{S}^d$, $0 \leq r \leq 2$, and

$$\gamma_d := \frac{\Gamma\left(\frac{d+1}{2}\right)}{\Gamma(d/2)\Gamma(1/2)}, \tag{2.5}$$

there holds

$$\sigma_d(r) := \sigma_d(B(x, r)) = \gamma_d \int_{1-r^2/2}^{1} (1 - t^2)^{d/2-1} dt$$

from which it follows that
$$\sigma_d(r) \le \frac{\gamma_d}{d} r^d,$$
and, as $r \to 0^+$,
$$\sigma_d(r) = \frac{\gamma_d}{d} r^d + \mathcal{O}(r^{d+2}).$$
Therefore, for the uniform probability distribution on $\mathbb{S}^d$, the global upper regularity constant is
$$C_0 = \sup_{0<r\le 2} \frac{\sigma_d(r)}{r^d} = \frac{\gamma_d}{d}, \tag{2.6}$$
and when applied to (2.4) we obtain
$$C_2 \ge \left(\frac{d}{\gamma_d}\right)^{1/d} \left(1 - \frac{d}{s}\right)^{1/d} \left(\frac{d}{s}\right)^{1/s}. \tag{2.7}$$
With this lower bound for $C_2$, (2.2) becomes
$$\delta(\omega_N^*) \ge \left(\frac{d}{\gamma_d}\right)^{1/d} \left(1 - \frac{d}{s}\right)^{1/d} \left(\frac{d}{s}\right)^{1/s} N^{-1/d} \qquad (N \ge 2,\ s > d), \tag{2.8}$$
and, on letting $s \to \infty$, we deduce for the $N$-point best-packing distance
$$\delta_N(\mathbb{S}^d) \ge \left(\frac{d}{\gamma_d}\right)^{1/d} N^{-1/d} \qquad (N \ge 2,\ s > d).$$
A less explicit lower bound for the separation constant of minimal energy points for $s > d$ on $\mathbb{S}^d$ was obtained in [13, Corollary 4].

We next consider the mesh norm of $(s, w)$-energy minimizing configurations on an $\alpha$-regular compact metric space $A$. In this case we require that the weight function $w$ be bounded.

**Theorem 3.** *Let $A$ be a compact, $\alpha$-regular metric space with respect to the measure $\mu$ and $K \subset A$ be a compact set of positive $\mu$-measure. Let $w$ be a bounded SLP weight on $K$. If $s > \alpha$ and $\omega_N^*$ is an $N$-point $(s, w)$-energy minimizing configuration on $K$, then*
$$\rho(\omega_N^*, K) \le C_3\, N^{-1/\alpha} \qquad (N \ge 2), \tag{2.9}$$
*where $C_3$ is a constant independent of $N$ given below in* (3.41).

Theorem 3 substantially extends a result of [6] that holds for unweighted energy minimizing point configurations when $K \subset \mathbb{R}^p$ is restricted to be the finite union of bi-Lipschitz images of compact sets in $\mathbb{R}^d$.

We remark that for $K$ and $A$ as in Theorem 3, the set $K$ need not inherit the lower $\alpha$-regularity of $A$. However, since $\mu(K) > 0$, we do have that $K$ is an upper $\alpha$-regular metric space and, consequently, there is a constant $\tilde{c}_K > 0$ such that (1.10) holds with $A$ replaced by $K$. Hence, the inequality (2.9) has the best possible order with respect to $N$.

Taking $w \equiv 1$ in Theorem 3 immediately yields the following.

**Corollary 4.** *Let $A$ be a compact, $\alpha$-regular metric space with respect to the measure $\mu$ and let $K \subset A$ be a compact set of positive $\mu$-measure. Then there exists a constant $C_4$ such that*
$$\rho_N(K) \le C_4\, N^{-1/\alpha} \qquad (N \ge 2).$$

Combining Corollary 2 and Theorem 3 we obtain our main result.

**Theorem 5.** *Let $A$ be a compact, $\alpha$-regular metric space with respect to the measure $\mu$ and let $K \subset A$ be a compact set of positive $\mu$-measure. Furthermore, let $w$ be a bounded SLP weight on $K$, and for $s > \alpha$ and $N \geq 2$, let $\omega_N^*$ be an $N$-point $(s, w)$-energy minimizing configuration on $K$. Then $\{\omega_N^*\}_{N=2}^{\infty}$ is quasi-uniform on $K$.*

We remark that there are $\alpha$-regular sets $A$ and values of $s < \alpha$ for which (unweighted) $(s, 1)$-energy minimizing configurations on $A$ have a mesh-separation ratio that goes to $\infty$ with $N$. One such example given in [4] is a 'washer' $A$ obtained by revolving a certain rectangle about an axis parallel to one of its sides, where it turns out that for $s < 1/3$, the support of the limit distribution of the $(s, 1)$-energy minimizing configurations on $A$ omits an open subset of $A$. Also, for the logarithmic energy which corresponds to $s = 0$, it is shown in [11] that, for $w \equiv 1$, the support of the limit distribution of the log-energy minimizing configurations on a torus in $\mathbb{R}^3$ is only supported on the positive curvature portion of the torus, so that the mesh-separation ratio for such configurations is again unbounded as $N \to \infty$. Examples also abound in one dimension. For the logarithmic energy, it is well-known [21, Sections 6.7 and 6.21] that for $A = [-1, 1]$ and $w \equiv 1$ the minimum energy points are zeros of Jacobi orthogonal polynomials (together with $\pm 1$) that have separation distance of precise order $1/N^2$ and mesh norm of precise order $1/N$, so that the mesh-separation ratio grows like $N$.

One of our main motivations for considering weighted minimum energy configurations is that for a large class of sets $A$ one can design a weight function $w$ so that a sequence of $N$-point $(s, w)$-energy minimizing configurations have a specified limiting density on $A$ as $N \to \infty$. The following result is a consequence of Theorem 5 and [2, Corollary 2]. Recall that a set in $\mathbb{R}^p$ is *$d$-rectifiable* if it is the Lipschitz image of a bounded set in $\mathbb{R}^d$.

**Corollary 6.** *Let $d \leq p$ and $A \subset \mathbb{R}^p$ be a compact, infinite set that is $d$-rectifiable and lower $d$-regular with respect to $\mathcal{H}_d$ for some integer $d$. Suppose $\sigma$ is a probability density on $A$ that is continuous almost everywhere with respect to $\mathcal{H}_d$ and is bounded above and below by positive constants. Let $s > d$ and $w : A \times A \to [0, \infty)$ be given by*

$$w(x, y) := (\sigma(x)\sigma(y))^{-s/2d}. \tag{2.10}$$

*For $N \geq 2$, let $\omega_N^*$ be an $N$-point $(s, w)$-energy minimizing configuration on $A$. Then $\{\omega_N^*\}_{N=2}^{\infty}$ is quasi-uniform on $A$ and the sequence of normalized counting measures associated with the $\omega_N^*$'s converges weak-star (as $N \to \infty$) to $\sigma \, \mathrm{d}\mathcal{H}_d$.*

For $A$ an infinite, compact, metric space and $s > 0$, let $\omega_N^s$ be an $N$-point $(s, 1)$-energy minimizing configuration on $A$. Furthermore, let $\nu_N$ be a cluster point (in the product topology on $A^N$) of $\omega_N^s$ as $s \to \infty$. As we now show, $\nu_N$ must be an *$N$-point best-packing configuration on $A$*, that is, $\delta(\nu_N) = \delta_N(A)$. For this purpose, let $\tilde{\omega}_N$ be an $N$-point best-packing configuration on $A$. Then we have

$$\delta(\omega_N^s)^{-s} \leq \mathcal{E}_s^1(N, A) \leq E_s^1(\tilde{\omega}_N) \leq N(N-1)\delta_N(A)^{-s},$$

and so

$$(N(N-1))^{-1/s}\delta_N(A) \leq \delta(\omega_N^s) \leq \delta_N(A),$$

which gives

$$\lim_{s\to\infty} \delta(\omega_N^s) = \delta_N(A). \tag{2.11}$$

Since $\omega_N^{s_j} \to \nu_N$ for some subsequence $s_j \to \infty$, it follows from (2.11) and continuity that $\delta(\nu_N) = \delta_N(A)$ and so $\nu_N$ is an $N$-point best-packing configuration on $A$.

In general, it is not true that a sequence of $N$-point best-packing configurations in $A$ is quasi-uniform on $A$ (e.g., if $A$ is the classical (1/3)-Cantor set in [0,1] together with any point outside this interval). However, for $A$ as in Theorem 5, it turns out that by using $(s,1)$-energy minimizing configurations on $A$ and taking $s \to \infty$ we can construct a sequence of $N$-point best-packing configurations in $A$ that is also quasi-uniform on $A$.

**Theorem 7.** *Let $A$ be a compact, $\alpha$-regular metric space with respect to the measure $\mu$ and let $K \subset A$ be a compact set of positive $\mu$-measure. For $N \geq 2$, let $\nu_N$ be a cluster point of a family of $N$-point $(s,1)$-energy minimizing configurations on $K$ as $s \to \infty$. Then $\{\nu_N\}_{N=2}^{\infty}$ is a sequence of $N$-point best-packing configurations on $K$ that is also quasi-uniform on $K$.*

*Furthermore, the mesh-separation ratios satisfy*

$$\limsup_{N\to\infty} \gamma(\nu_N, K) \leq\ 2\left(\frac{\mu(A)}{\mu(K)}\right)^{1/\alpha} [c_0(0)\, C_0(0)]^{1/\alpha}, \tag{2.12}$$

*where $c_0(0)$ and $C_0(0)$ are given in (1.8) for the set $A$.**

We note that the constant on the right-hand side of (2.12) is at least 2 per (1.7) and (1.8). One can also establish an analogous result concerning the existence of quasi-uniform sequences of *weighted* best-packing configurations (cf. [3]). We leave this extension to the reader.

In comparison with (2.12), we remark that one can construct examples of metric spaces $A$ having $n$-point best-packing configurations with arbitrarily large mesh-separation ratio.

We conclude this section with further references to related results. Separation theorems for the case $s \leq d = \dim_{\mathcal{H}}(A)$ have been established only for rather special sets and values of $s$. Dahlberg [5] proved that (unweighted) optimal $((p-2),1)$-energy configurations $\omega_N^*$ on $A$ are *well-separated* (i.e., they satisfy $\delta(\omega_N^*) \geq CN^{-1/d}$ for some positive constant $C$) if $A \subset \mathbb{R}^p$ $(p \geq 3)$ is a smooth $d = p-1$ dimensional closed surface in $\mathbb{R}^p$ that separates $\mathbb{R}^p$ into two components. For the critical value $s = d$ and $A$ a $d$-rectifiable subset of a smooth $d$-dimensional manifold in $\mathbb{R}^p$, it is shown in [2] that the following weaker separation result holds

$$\delta(\omega_N^*) \geq C(N\log N)^{-1/d}, \tag{2.13}$$

for some positive constant $C$.

For the case that $A = \mathbb{S}^d$, the $d$-dimensional unit sphere in $\mathbb{R}^{d+1}$, well-separation was proved in [14] for the range of values $d-1 < s < d$ and further extended by Dragnev and Saff [8] to the range $d-2 < s < d$ with explicit estimates for the

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**Added in proof:* In the manuscript [1], the first two authors together with A. Bondarenko have recently proved under more general conditions that the right-hand side of (2.12) can be replaced by 1.

separation constant $C$. Well-separation for $s = d-2$ and $d \geq 3$ was established in [6].

Thus, for the important case of $A = \mathbb{S}^2$ it is known that optimal $s$-energy configurations on $\mathbb{S}^2$ are well-separated for all nonnegative values of $s \neq 2$ (well-separatedness for $s = 0$ was established in [18]; see also [7]); for the critical value $s = 2$, the only known separation results are of the weak form given in (2.13).

Much less is known with regard to covering (mesh norm) theorems in the case that $s \leq d$ (see [20, Sec. 1.3]).

## 3. Proofs

In the proofs we shall need that an SLP weight $w$ is bounded below in a neighborhood of the diagonal $D(A)$. Indeed, the positivity and lower semi-continuity of $w$ on $D(A)$ and the compactness of $A$ imply that there are positive numbers $\eta$ and $\kappa$ such that

$$w(x,y) \geq \eta \qquad (x, y \in A,\, m(x,y) \leq \kappa). \tag{3.1}$$

*Proof of Theorem 1.* The initial part of this argument proceeds as in [13]. Let $N \geq 2$ be fixed and let $\omega_N^* = \{x_1, \dots, x_N\} \subset A$ be a fixed $(s,w)$-energy minimizing configuration in $A$. For $x \in A$ and $1 \leq i \leq N$, let

$$U_i(x) := \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{w(x, x_j)}{(x, x_j)^s}.$$

Since $\omega_N^*$ is a minimizing configuration we have the lower bound

$$U_i(x_i) \leq U_i(x) \text{ for all } x \in A. \tag{3.2}$$

Fix $r_1 \leq \operatorname{diam}(A)$ such that

$$\bar{\mu}\left(\bigcup_{j=1}^{N} B(x_j, r_1)\right) \geq \bar{\mu}(A). \tag{3.3}$$

The radius $r_1$ can clearly be chosen independent of $N$, for example $r_1 = \operatorname{diam}(A)$, and we note for future reference that it suffices to take $r_1 > \rho(\omega_N^*, A)$. For the rest of this proof we fix $r_1 = \operatorname{diam}(A)$.

Now let $0 < \theta < 1$ and define

$$r_0 := \left(\frac{\theta \bar{\mu}(A)}{N\, C_0(r_1)}\right)^{1/\alpha}, \tag{3.4}$$

where $C_0(r_1) = C_0$ is the upper regularity constant of $\bar{\mu}$ as in (1.7). We note that $r_0 < r_1$ as can be seen from the fact that $\bar{\mu}(A) \leq C_0(r_1) r_1^{\alpha}$.

For $B(x, r_0, r_1) := B(x, r_1) \setminus B(x, r_0)$, let

$$D := \bigcup_{j=1}^{N} B(x_j, r_0, r_1).$$

Using the upper regularity of $\bar{\mu}$ and (3.3) we see that

$$\bar{\mu}(D) \geq \bar{\mu}(A) - \sum_{j=1}^{N} \bar{\mu}(B(x_j, r_0)) \geq (1-\theta)\bar{\mu}(A) > 0,$$

and thus by inequality (3.2) we have

$$U_i(x_i) \leq \frac{1}{\bar{\mu}(D)} \int_D U_i(x)\, d\bar{\mu}(x) \leq \frac{1}{(1-\theta)\bar{\mu}(A)} \sum_{\substack{j=1 \\ j\neq i}}^{N} \int_{B(x_j, r_0, r_1)} \frac{w(x, x_j)}{\dot{(x, x_j)}^s}\, d\bar{\mu}(x). \tag{3.5}$$

Applying Hölder's inequality with $1/q = 1 - 1/p$ we obtain

$$U_i(x_i) \leq \frac{1}{(1-\theta)\bar{\mu}(A)} \sum_{\substack{j=1 \\ j\neq i}}^{N} \|w(\cdot, x_j)\|_{L_p(\bar{\mu})} \left( \int_{B(x_j, r_0, r_1)} \frac{1}{\dot{(x, x_j)}^{sq}}\, d\bar{\mu}(x) \right)^{1/q}. \tag{3.6}$$

Converting the integral on the right-hand side of (3.6) to the appropriate integral of the distribution function, and noting that $sq > \alpha$ by assumption, we have

$$\begin{aligned}
\int_{B(x_j, r_0, r_1)} \frac{1}{\dot{(x, x_j)}^{sq}}\, d\bar{\mu}(x) &= \int_0^\infty \bar{\mu}\left(\{x \in B(x_j, r_0, r_1) : m(x_j, x)^{-sq} > t\}\right) dt \\
&\leq \int_{r_1^{-sq}}^{r_0^{-sq}} \bar{\mu}\left(B(x_j, t^{-1/sq})\right) dt \\
&\leq \frac{C_0(r_1)\, sq}{sq - \alpha}\, r_0^{\alpha - sq} \\
&= \frac{C_0(r_1)\, sq}{sq - \alpha} \left( \frac{\theta \bar{\mu}(A)}{N\, C_0(r_1)} \right)^{1-(sq)/\alpha},
\end{aligned} \tag{3.7}$$

which, combined with (3.6), gives

$$\begin{aligned}
U_i(x_i) &\leq \frac{\|w\|_{p,\infty}}{(1-\theta)\bar{\mu}(A)} \left( \frac{C_0(r_1)\, sq}{sq - \alpha} \right)^{1/q} (N-1) \left( \frac{\theta \bar{\mu}(A)}{N\, C_0(r_1)} \right)^{1/q - s/\alpha} \\
&< \frac{1}{\bar{\mu}(A)} \left( \frac{C_0(r_1)}{\bar{\mu}(A)} \right)^{s/\alpha} \left( \frac{\|w\|_{p,\infty}}{(1-\theta)\theta^{s/\alpha - 1/q}} \right) \left( \frac{sq\bar{\mu}(A)}{sq - \alpha} \right)^{1/q} N^{1/p + s/\alpha},
\end{aligned} \tag{3.8}$$

where $\|w\|_{p,\infty} := \sup_{x\in A} \|w(\cdot, x)\|_{L_p(\bar{\mu})} < \infty$.

Choosing

$$\theta_0 := \frac{sq - \alpha}{sq - \alpha + \alpha q} = \left( \frac{s}{\alpha} - \frac{1}{q} \right) \left( \frac{s}{\alpha} + \frac{1}{p} \right)^{-1} < 1, \tag{3.9}$$

which minimizes the right-hand side of (3.8) with respect to $\theta$, we obtain

$$U_i(x_i) \leq c_1 N^{s/\alpha + 1/p}, \tag{3.10}$$

where after a bit of arithmetic we have

$$c_1 := \|w\|_{p,\infty} \left( \frac{C_0(r_1)}{\bar{\mu}(A)} \frac{s/\alpha + 1/p}{s/\alpha - 1/q} \right)^{s/\alpha} \left( \frac{s/\alpha + 1/p}{\bar{\mu}(A)} \right)^{1/p} (s/\alpha)^{1/q}. \tag{3.11}$$

Next, select the indices $1 \le i_s \ne j_s \le N$ so that $\delta(\omega_N^*) = (x_{i_s}, x_{j_s})$ and let $\kappa$ and $\eta$ be as in (3.1). If $\delta(\omega_N^*) \le \kappa$, then

$$\frac{\eta}{\delta(\omega_N^*)^s} \le \frac{w(x_{i_s}, x_{j_s})}{(x_{i_s}, x_{j_s})^s} \le U_{i_s}(x_{i_s}) \le c_1 N^{s/\alpha + 1/p}, \tag{3.12}$$

and therefore

$$\delta(\omega_N^*) \ge \left(\frac{\eta}{c_1}\right)^{1/s} N^{-\frac{1}{\alpha} - \frac{1}{sp}}.$$

Hence, (2.1) holds with

$$C_1 := \min\{\kappa,\ (\eta/c_1)^{1/s}\}. \tag{3.13}$$

□

We remark that for the case when $w \equiv 1$ and $p = \infty$, we can take $\kappa = \infty$, $\eta = 1$, and so from (3.13) we deduce the separation estimate

$$\delta(\omega_N^*) \ge C_2 N^{-1/\alpha} \qquad (N \ge 2),$$

where

$$C_2 := \left[\frac{\bar{\mu}(A)}{C_0(r_1)}(1 - \alpha/s)\right]^{1/\alpha} (\alpha/s)^{1/s},\ r_1 = \operatorname{diam}(A). \tag{3.14}$$

For the proof of Theorem 3, we utilize the following.

**Lemma 8.** *Let $A$ be a compact, infinite, lower $\alpha$-regular metric space with lower $\alpha$-regular measure $\underline{\mu}$, $w : A \times A \to [0, \infty)$ be an SLP weight on $A$, and $s > \alpha$. Then there exists a positive integer $N_0$ independent of $s$, such that*

$$\mathcal{E}_s^w(N, A) \ge C_5 N^{1+s/\alpha} \qquad (N \ge N_0), \tag{3.15}$$

*where $C_5$ is a constant independent of $N$ given below in* (3.19).

*Proof.* Let $\kappa$ and $\eta$ be as in (3.1) and let $0 < r_2 \le \kappa$. Since $A$ is compact, there is some $M$ such that the $M$-point best-packing distance satisfies

$$\delta_M(A) \le r_2. \tag{3.16}$$

Let $N > M$ and let $\omega_N = \{x_1, \ldots, x_N\} \subset A$ be an arbitrary $N$-point configuration of distinct points. For $1 \le i \le N$, let $y_i \in \omega_N$ be a fixed nearest neighbor to $x_i$ in the configuration $\omega_N$, and set

$$\delta_i := (x_i, y_i) = \min_{\substack{1 \le j \le N \\ j \ne i}} (x_i, x_j) > 0.$$

We assume an ordering on $\omega_N$ so that $\delta_i \le \delta_{i+1}$ for $i = 1, \ldots, N-1$. We note that $\omega_N \setminus \{x_1, \ldots, x_{N-M}\}$ is of cardinality $M$ and thus for all $i \le N' := N - M$ we have that $\delta_i \le r_2 \le \kappa$.

The energy of $\omega_N$ then has the lower bound

$$\begin{aligned} E_s^w(\omega_N) &\ge \sum_{i=1}^{N'} \frac{w(x_i, y_i)}{\delta_i^s} \ge \sum_{i=1}^{N'} \eta \left(\frac{1}{\delta_i^\alpha}\right)^{s/\alpha} \ge \eta \left(\sum_{i=1}^{N'} \frac{1}{\delta_i^\alpha}\right)^{s/\alpha} (N')^{1-s/\alpha} \\ &\ge \eta \left(\sum_{i=1}^{N'} \delta_i^\alpha\right)^{-s/\alpha} (N')^{1+s/\alpha} = \eta 2^{-s} \left(\sum_{i=1}^{N'} \left(\frac{\delta_i}{2}\right)^\alpha\right)^{-s/\alpha} (N')^{1+s/\alpha}. \end{aligned} \tag{3.17}$$

where the last inequality in the first line follows from Jensen's inequality and the subsequent inequality follows from the harmonic-arithmetic mean inequality.

Let $\Lambda > 1$ and $N_0 := M\Lambda/(\Lambda - 1)$. Then $N' = N - M \geq \Lambda^{-1}N$ for $N \geq N_0$. Noting that the balls $B(x_i, \delta_i/2)$ are pairwise disjoint, we may apply the lower regularity of $\underline{\mu}$ (with regularity constant $c_0(r_2)$) to obtain

$$\begin{aligned} E_s^w(\omega_N) &\geq \eta 2^{-s} \left( c_0(r_2) \sum_{i=1}^{N'} \underline{\mu} \left( B(x_i, \frac{\delta_i}{2}) \right) \right)^{-s/\alpha} (N')^{1+s/\alpha} \\ &\geq \frac{\eta}{(2^\alpha\, c_0(r_2)\, \underline{\mu}(A))^{s/\alpha}} (N')^{1+s/\alpha} \\ &\geq \Lambda^{-1-s/\alpha} \frac{\eta}{(2^\alpha\, c_0(r_2)\, \underline{\mu}(A))^{s/\alpha}} N^{1+s/\alpha} \end{aligned} \quad . \tag{3.18}$$

Since (3.18) holds for arbitrary $N$-point configurations $\omega_N \subset A$ with $N \geq N_0$, we obtain that (3.15) holds with

$$C_5 := \Lambda^{-1-s/\alpha}\, \eta\, 2^{-s}\, (c_0(r_2)\, \underline{\mu}(A))^{-s/\alpha}. \tag{3.19}$$

We remark that $N_0$ depends on $\Lambda$ and $r_2$, but is independent of $s$. □

*Proof of Theorem 3.* Appealing to the generality provided by Theorem 1 and Lemma 8, we can substantially extend and improve upon the arguments used in the proof of Theorem 3.6 in [6].

Let $\omega_N^* = \{x_1 \ldots, x_N\}$ be an $N$-point $(s, w)$-energy minimizing configuration for the compact set $K$, and, for $y \in K$, consider the function

$$U(y) := \frac{1}{N} \sum_{i=1}^{N} \frac{w(y, x_i)}{(y, x_i)^s}. \tag{3.20}$$

For fixed $1 \leq j \leq N$, the function $U(y)$ can be decomposed as

$$U(y) = \frac{1}{N} \frac{w(y, x_j)}{(y, x_j)^s} + \frac{1}{N} \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{w(y, x_i)}{(y, x_i)^s}, \tag{3.21}$$

and, since $\omega_N^*$ is a minimizing configuration on $K$, the point $x_j$ minimizes the sum over $i \neq j$ on the right-hand side of equation (3.21). Thus for each fixed $j$ and $y \in K$

$$U(y) \geq \frac{1}{N} \frac{w(y, x_j)}{(y, x_j)^s} + \frac{1}{N} \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{w(x_j, x_i)}{m(x_j, x_i)^s}. \tag{3.22}$$

Summing over $j$ gives

$$NU(y) \geq \frac{1}{N} \sum_{j=1}^{N} \frac{w(y, x_j)}{(y, x_j)^s} + \frac{1}{N} \sum_{j=1}^{N} \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{w(x_j, x_i)}{m(x_j, x_i)^s} \tag{3.23}$$

$$= U(y) + \frac{1}{N} \mathcal{E}_s^w(N, K), \tag{3.24}$$

and thus

$$U(y) \geq \frac{1}{N(N-1)} \mathcal{E}_s^w(N,K) \geq \frac{\mathcal{E}_s^w(N,K)}{N^2} \qquad (y \in K). \tag{3.25}$$

Since $K$ is compact, there exists a point $y^* \in K$ such that

$$\min_{1\leq i \leq N} (y^*, x_i) = \rho(\omega_N^*, K) =: \rho(\omega_N^*). \tag{3.26}$$

Using the fact that a function is lower semi-continuous if and only if it is the limit of an increasing sequence of continuous functions, it is not difficult to show that since $w$ is a bounded SLP weight on $K$, it may be extended to a bounded SLP weight on $A$. Then, by Lemma 8, there are constants $N_0$ and $C_5 > 0$ such that

$$\mathcal{E}_s^w(N,K) \geq \mathcal{E}_s^w(N,A) \geq C_5 N^{1+s/\alpha} \qquad (N \geq N_0). \tag{3.27}$$

We note that the constant $C_5$ of (3.27) does not depend on $K$, but rather on $A$ (specifically on the lower regularity constant of $A$ and on $\mu(A)$) as well as on the extended weight $w$.

Since (3.25) holds for the point $y^*$ of (3.26), we combine (3.25) with (3.27) to obtain

$$U(y^*) \geq \frac{\mathcal{E}_s^w(N,K)}{N^2} \geq C_5 N^{s/\alpha - 1} \qquad (N \geq N_0). \tag{3.28}$$

Next we determine an upper bound for $U(y^*)$ using the $\alpha$-regularity of the superset $A$. Since $A$ is upper $\alpha$-regular, we see that $K$ is also because $\mu(K) > 0$. Hence, Corollary 2 applied to $K$ implies that there is some $C_2 > 0$ such that $\delta(\omega_N^*) \geq C_2 N^{-1/\alpha}$ for $N \geq 2$. We note that the constant $C_2$ here depends on $K$, specifically $\mu(K)$.

Let $\mathcal{N}$ consist of those $N \geq N_0$ such that

$$\rho(\omega_N^*) \geq \frac{C_2}{2} N^{-1/\alpha}. \tag{3.29}$$

If $\mathcal{N}$ is empty (or finite) then we are done. Assuming that $\mathcal{N}$ is nonempty, let $N \in \mathcal{N}$ be fixed.

For $0 < \epsilon < 1/2$, let

$$r_0 = r_0(N, \epsilon) := \epsilon\, C_2 N^{-1/\alpha}. \tag{3.30}$$

Note that any two of the balls $B(x_i, r_0) \subset A$, for $1 \leq i \leq N$, do not intersect since $r_0 < \delta(\omega_N^*)/2$.

For any $x \in B(x_i, r_0)$, inequalities (3.26) and (3.29) imply

$$\begin{aligned} (x, y^*) &\leq (x, x_i) + (x_i, y^*) \leq r_0 + (x_i, y^*) \\ &\leq 2\epsilon\, \rho(\omega_N^*) + (x_i, y^*) \leq (1+2\epsilon)(x_i, y^*). \end{aligned} \tag{3.31}$$

For fixed $1 \leq i \leq N$, using (3.31) and taking an average value on $B(x_i, r_0)$ we obtain

$$\begin{aligned} \frac{w(x_i, y^*)}{(x_i, y^*)^s} &\leq \frac{\|w\|_\infty (1+2\epsilon)^s}{\mu(B(x_i, r_0))} \int_{B(x_i, r_0)} \frac{d\mu(x)}{(x, y^*)^s} \\ &\leq \frac{\|w\|_\infty\, (1+2\epsilon)^s\, c_0(r_0)}{r_0^\alpha} \int_{B(x_i, r_0)} \frac{d\mu(x)}{(x, y^*)^s}, \end{aligned} \tag{3.32}$$

where $\|w\|_\infty$ denotes the sup-norm of $w$ on $A\times A$ and $c_0(r_0)$ is the localized constant of (1.7) for the set $A$.

Inequality (3.29) and definition (3.30) imply $2\epsilon\rho(\omega_N^*) \geq r_0$ and thus, for $x \in B(x_i, r_0)$, we obtain

$$
\begin{aligned}
(x, y^*) &\geq (x_i, y^*) - (x, x_i) \geq (x_i, y^*) - r_0 \\
&\geq (x_i, y^*) - 2\epsilon\, \rho(\omega_N^*) \geq (1-2\epsilon)\rho(\omega_N^*).
\end{aligned}
\tag{3.33}
$$

Inequality (3.33) implies

$$
\bigcup_{i=1}^{N} B(x_i, r_0) \subset A \setminus B(y^*, (1-2\epsilon)\rho(\omega_N^*)),
$$

and since the left-hand side is a disjoint union, averaging the inequalities of (3.32) we have

$$
\begin{aligned}
U(y^*) &\leq \frac{\|w\|_\infty\, (1+2\epsilon)^s\, c_0(r_0)}{N\, r_0^\alpha} \sum_{i=1}^{N} \int_{B(x_i,r_0)} \frac{d\mu(x)}{(x, y^*)^s} \\
&\leq \frac{\|w\|_\infty\, (1+2\epsilon)^s\, c_0(r_0)}{N\, r_0^\alpha} \int_{A\setminus B(y^*,(1-2\epsilon)\rho(\omega_N^*))} \frac{d\mu(x)}{(x, y^*)^s}.
\end{aligned}
\tag{3.34}
$$

For fixed $\tau \geq 1$ we define the radius $R(N) := \tau(1-2\epsilon)\rho(\omega_N^*)$, and the constant

$$
\tilde{C}_0(\tau) := C_0(R(N))(1-\tau^{\alpha-s}) + C_0\tau^{\alpha-s}. \tag{3.35}
$$

Note that if $\tau = 1$, then $\tilde{C}_0(1) = C_0$. (We retain $\tau$ as a parameter in our estimates as an option for the reader to optimize $C_3$ for a fixed $s$.) Now we break the integral on the right-hand side of (3.34) into two terms and proceed as in (3.7) to obtain

(3.36)

$$
\begin{aligned}
&\int_{A\setminus B(y^*,(1-2\epsilon)\rho(\omega_N^*))} \frac{d\mu(x)}{(x, y^*)^s} \\
&\qquad = \int_{B(y^*,(1-2\epsilon)\rho(\omega_N^*),R(N))} \frac{d\mu(x)}{(x, y^*)^s} + \int_{A\setminus B(y^*,R(N))} \frac{d\mu(x)}{(x, y^*)^s} \\
&\qquad \leq C_0(R(N)) \int_{R(N)^{-s}}^{[(1-2\epsilon)\rho(\omega_N^*)]^{-s}} t^{-\alpha/s} dt + C_0 \int_0^{R(N)^{-s}} t^{-\alpha/s} dt \\
&\qquad = \frac{\tilde{C}_0(\tau)}{(1-\alpha/s)(1-2\epsilon)^{s-\alpha}} \rho(\omega_N^*)^{\alpha-s}.
\end{aligned}
$$

It is convenient to define the quantity

$$
\beta(\epsilon) := \frac{\|w\|_\infty (1+2\epsilon)^s}{(1-\alpha/s)(1-2\epsilon)^{s-\alpha}(\epsilon C_2)^\alpha}, \tag{3.37}
$$

and we note that for fixed $s > \alpha$ it is minimized as a function of $\epsilon$ for

$$
\epsilon_0 := \frac{1}{2(2(s/\alpha)-1)} < \frac{1}{2}, \tag{3.38}
$$

with minimal value

$$
\beta_0 := \beta(\epsilon_0) = \frac{\|w\|_\infty}{(1-\alpha/s)^{s-\alpha+1}} \left(\frac{4s}{\alpha C_2}\right)^\alpha. \tag{3.39}
$$

Using $\epsilon_0$ and combining inequality (3.34) with inequality (3.36) we obtain

$$U(y^*) \le c_0(r_0)\beta_0\tilde{C}_0(\tau)\rho(\omega_N^*)^{\alpha-s}. \tag{3.40}$$

If $N \in \mathcal{N}$, then (3.40) and (3.28) imply

$$\rho(\omega_N^*) \le \left[\frac{c_0(r_0)\beta_0\tilde{C}_0(\tau)}{C_5}\right]^{1/(s-\alpha)} N^{-1/\alpha}.$$

If $N \notin \mathcal{N}$, then either $N \le N_0$ or $\rho(\omega_N^*) < \frac{C_2}{2}N^{-1/\alpha}$. Hence (2.9) holds with

$$C_3 := \max\left\{\operatorname{diam}(A)N_0^{1/\alpha},\ \left[\frac{c_0(r_0)\beta_0\tilde{C}_0(\tau)}{C_5}\right]^{1/(s-\alpha)}, \frac{C_2}{2}\right\}. \tag{3.41}$$

We note that if $N > N_0$, then it suffices to take

$$C_3 = \max\left\{\left[\frac{c_0(r_0)\beta_0\tilde{C}_0(\tau)}{C_5}\right]^{1/(s-\alpha)}, \frac{C_2}{2}\right\} \tag{3.42}$$

□

*Proof of Theorem 7.* Starting with Theorem 3 we shall employ a bootstrapping argument whereby the constants $C_2$, $C_5$, and subsequently $C_3$ are redefined so as to depend on $N$.

We begin by noting that if $s \ge 2\alpha$, then the constant $C_3$ of (3.41) has a uniform upper bound in $s$; indeed, with $\kappa = \infty, C_2$ as defined in (3.14) and $C_5$ as defined in (3.19) (with $\eta = 1$), each of the three terms appearing in braces in (3.41) is uniformly bounded above. Thus there exists a constant $C^*$ independent of $N \ge 2$ and of $s \ge 2\alpha$ such that $\rho(\omega_N^{(s)}, K) < C^* N^{-1/\alpha}$, where $\omega_N^{(s)}$ is any $N$-point $(s,1)$-energy minimizing configuration on $K$.

We next note that $C_0(0)$ of (1.8) is finite and positive, and utilizing the constant $c_A$ of (1.9) we fix

$$C^{**} := \max\left\{C^*,\ c_A,\ \left(\frac{\mu(K)}{C_0(0)}\right)^{1/\alpha}\right\}, \tag{3.43}$$

and we now redefine the radius $r_1$ to be a function of $N$,

$$r_1(N) := C^{**}\, N^{-1/\alpha} \quad (N \ge 2). \tag{3.44}$$

Returning to the proof of Theorem 1, we note that $r_1(N) > \rho(\omega_N^{(s)}, K)$, and so inequality (3.3) holds. Furthermore, by the choice of $C^{**}$ we have that for $0 < \theta_0 < 1$ as in (3.9)

$$r_0(N) := \left(\frac{\theta_0\mu(K)}{NC_0(0)}\right)^{1/\alpha} < r_1(N).$$

Taking $r_0 = r_0(N)$ in the proof and remembering that $q = 1$ in the current context, we see that with $A$ replaced by $K$ the penultimate term on right-hand side of (3.7) becomes

$$\frac{sC_0(r_1(N))}{s-\alpha}\left(\frac{\theta_0\,\mu(K)}{N\,C_0(0)}\right)^{1-s/\alpha},$$

and thus

$$\begin{aligned}\int_{B(x_j,r_0(N),r_1(N))}\frac{d\mu(x)}{(x,x_j)^s} &\le \frac{sC_0(r_1(N))}{s-\alpha}\left(\frac{\theta_0\,\mu(K)}{N\,C_0(0)}\right)^{1-s/\alpha}\\ &\le \frac{s}{s-\alpha}\left(\frac{\theta_0\,\mu(K)}{N}\right)^{1-s/\alpha} C_0(r_1(N))^{s/\alpha},\end{aligned} \tag{3.45}$$

where the last inequality follows from the fact that $C_0(0)\le C_0(r_1(N))$ and $s>\alpha$.

For $w\equiv 1$, the constant $C_2$ of (3.14) with $r_1=r_1(N)$ becomes

$$C_2(N) := \left(\frac{\alpha}{s}\right)^{1/s}\left(\frac{1-\alpha/s}{C_0(r_1(N))}\right)^{1/\alpha}\mu(K)^{1/\alpha}, \tag{3.46}$$

where $C_0(r_1(N))$ is the local upper regularity constant of (1.7), and we have

$$\delta(\omega_N^{(s)})\ge C_2(N)N^{-1/\alpha}\qquad (N\ge 2,\ s\ge 2\alpha).$$

Furthermore, allowing the radius $r_2$ appearing in (3.16) to depend on $N\ge 2$ by taking $r_2 := r_1(N)$, we see via (1.9) and (3.43) that

$$r_1(N)\ge \delta_N(A)\quad (N\ge 2),$$

and there is no need to designate the integer $M$ in the proof of Lemma 8. Thus we can take $\Lambda=1$ in (3.19), and it follows (with $\eta=1$) that

$$E_s^1(\omega_N^{(s)})\ge C_5(N)N^{1+s/\alpha}\qquad (N\ge 2,\ \ s\ge 2\alpha),$$

where

$$C_5(N) := \frac{1}{2^s[c_0(r_1(N))\mu(A)]^{s/\alpha}}. \tag{3.47}$$

We remark that $C_2(N)$ clearly depends on the subset $K$, whereas $C_5(N)$ depends on the superset $A$.

We now return to the proof of Theorem 3 utilizing the constants $C_2(N)$ and $C_5(N)$. For $\beta_0$ as in (3.39), we see that

$$\rho(\omega_N^{(s)},K)\le C_3(N)N^{-1/\alpha}\ \ (N\ge N_0,\ \ s\ge 2\alpha),$$

where $N_0$ is as in Lemma 8, and by (3.42) (choosing $\tau=1$, so that $\tilde{C}_0(\tau)=C_0$)

$$C_3(N) := \max\left\{\left[\frac{c_0(r_0)\beta_0 C_0}{C_5(N)}\right]^{1/(s-\alpha)}, \frac{C_2(N)}{2}\right\}. \tag{3.48}$$

With equations (3.46)-(3.48) in mind, we are ready to complete the proof of Theorem 7. The argument leading to equation (2.11) shows that $\nu_N$ is an $N$-point best-packing configuration on $K$ for each $N\ge 2$. We now need to determine the limits of the constants $C_2(N)$ of (3.46) and $C_3(N)$ of (3.48) as $s\to\infty$. Fixing $N$ in (3.46) yields

$$\lim_{s\to\infty} C_2(N) = \left(\frac{\mu(K)}{C_0(r_1(N))}\right)^{1/\alpha} =: \hat{C}_2(N). \tag{3.49}$$

Since $c_0(r_0)$ and $C_0$ are independent of $s$ and $\lim_{s\to\infty}\beta_0^{1/(s-\alpha)}=1$, it follows, that for fixed $N$

$$\begin{aligned}\lim_{s\to\infty} C_3(N) &= \max\left\{\frac{\hat{C}_2(N)}{2},\ \lim_{s\to\infty} C_5(N)^{1/(\alpha-s)}\right\}\\ &= \max\left\{\frac{1}{2}\left(\frac{\mu(K)}{C_0(r_1(N))}\right)^{1/\alpha},\ 2[c_0(r_1(N))\mu(A)]^{1/\alpha}\right\}.\\ &:= \hat{C}_3(N)\end{aligned}\tag{3.50}$$

From the continuity of $\delta(\cdot)$ and $\rho(\cdot,K)$ on $K^N$ we deduce that

$$\delta(\nu_N)\geq \hat{C}_2(N)N^{-1/\alpha}\quad\text{and}\quad \rho(\nu_N,K)\leq \hat{C}_3(N)N^{-1/\alpha}\qquad (N\geq N_0).$$

Taking the ratio of these two quantities we have that

$$\frac{\rho(\nu_N,K)}{\delta(\nu_N)}\leq\frac{\hat{C}_3(N)}{\hat{C}_2(N)}=\max\left\{\frac{1}{2},\ 2\left(\frac{\mu(A)}{\mu(K)}\right)^{1/\alpha}[c_0(r_1(N))\,C_0(r_1(N))]^{1/\alpha}\right\},\tag{3.51}$$

and hence for $N\geq N_0$

$$\limsup_{N\to\infty}\frac{\rho(\nu_N,K)}{\delta(\nu_N)}\leq\max\left\{\frac{1}{2},\ 2\left(\frac{\mu(A)}{\mu(K)}\right)^{1/\alpha}[c_0(0)\,C_0(0)]^{1/\alpha}\right\}\tag{3.52}$$

$$=2\left(\frac{\mu(A)}{\mu(K)}\right)^{1/\alpha}[c_0(0)\,C_0(0)]^{1/\alpha}<\infty.\tag{3.53}$$

Therefore, the sequence of configurations $\{\nu_N\}_{N=2}^{\infty}$ is quasi-uniform on $K$. $\square$

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